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Quantum Field Theory for a System of Interacting Photons, Electrons, and Phonons

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A Lagrangian in (1 + 3)-dimensional space-time which describes the interaction of photons, electrons, and phonons is proposed. This is a generalization of Rodriguez-Nuñez' model. This Lagrangian is also singular in the sense of Dirac. The path-integral quantization of this system is performed with the aid of the Dirac formalism for a singular Lagrangian and the method of functional integration. The phase-space generating functional of the Green function of this system is deduced. The Ward identities in canonical formalism for local symmetries are derived, and the Ward identities of proper vertices for this system are obtained. The conserved charges at the quantum level are also obtained. The effective Lagrangian in configuration space for the present model is derived in the case ρ = const. Thus, the Feynman rule can be deduced immediately.

1. INTRODUCTION

The electron–phonon system (polaron) is basic to the BCS theory of superconductivity for metals. This interaction has been expressed as a Hamiltonian (Haken, 1976) and described by using a Lagrangian in (1 + 1)-dimensional space-time (Rodriguez-Nuñez, 1990). The Lagrangian is singular in the sense of Dirac (1964). The canonical quantization by using Dirac brackets for this Lagrangian was given by Rodriguez-Nuñez (1990). Symmetry Dirac brackets were used to obtain the BCS Hamiltonian.

However, in that work the electromagnetic field was not included. In the present paper, we shall discuss a more general case. A Lagrangian which describes photon–electron–phonon interactions in (1 + 3)-dimensional space-time is proposed. This is a generalization of Rodriguez-Nuñez' (1990) model. This generalized Lagrangian is also singular in the sense of Dirac. We formu-

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late the path-integral quantization of this system with aid of the Dirac formalism for a singular Lagrangian and the method of functional integration (Faddeev, 1970; Senjanovic, 1976). The local and global canonical symmetry properties at the quantum level for a system with singular Lagrangian are discussed, and we give a preliminary application to the present model.

This paper is organized as follows. In Section 2 we propose a Lagrangian in (1 + 3)-dimensional space-time which describes photon–electron–phonon interactions, and formulate a path-integral quantization for this system. This system contains both first-class constraints and second-class constraints in the Dirac formalism. The generating functional of the Green function in phase space for this system is deduced. In Section 3 the canonical Ward identities for local symmetries are derived, and the Ward identities of proper vertices for a system of interacting photons, electrons, and phonons are also derived; the advantage of this derivation is that one does not need to carry out explicit integration over canonical momenta in the phase-space functional integral. In Section 4 the global canonical symmetry properties for the functional integral in the canonical formalism are studied. The conserved charges of space-time symmetries at the quantum level for a system of interacting photons, electrons, and phonons are obtained. In Section 5 the effective Lagrangian in configuration space is derived in the case ρ = const for the present model; thus the Feynman rule for this system can be derived immediately. Section 6 is devoted to conclusions and a discussion.

2. THE GENERATING FUNCTIONAL OF THE GREEN FUNCTION FOR A SYSTEM OF INTERACTING PHOTONS, ELECTRONS, AND PHONONS

The Lagrangian used to describe the electron–phonon interaction in (1 + 1)-dimensional space-time was given by Rodriguez-Nuñez (1990). The electromagnetic field was not included. Here we study a more general case; the electromagnetic field is considered and studied in the (1 + 3)-dimensional space-time for such a system. The phonon, electron, and electromagnetic fields are denoted by $q(x)$, $\psi(x)$, and $A_\mu(x)$ respectively, where $x = (t, \mathbf{x})$. The flat space-time metric is $\eta_{\mu\nu} = \text{diag}(+ - - -)$ ($\mu, \nu = 0, 1, 2, 3$). Natural units ($c = \hbar = 1$) are adopted. Throughout this paper the same notations as in Rodriguez-Nuñez (1990) will be used, unless otherwise stated. The inclusion of the electromagnetic field in the Lagrangian for a system of interacting photons, electrons, and phonons can be done by requiring that $\partial_\mu \to \partial_\mu - ieA_\mu$; thus, the Lagrangian density for such a system in (1 + 3)-dimensional space-time can be generalized as

$$\mathcal{L} = \frac{-1}{4} F_{\mu\nu}(x)F^{\mu\nu}(x) + \psi^*(x)\left[i(\partial_0 - ieA_0) + \frac{1}{2m}(\nabla - ie\mathbf{A})^2\right]\psi(x)$$

$$+ \frac{1}{2}[\rho(\dot{q}(x))^2 - s(\nabla q(x))^2] - V(x, \mathbf{q})\psi^*(x)\psi(x) \tag{1}$$

where $F_{\mu\nu}(x)$ is the usual electromagnetic tensor

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \tag{2}$$

$A_\mu(x) = (A_0(x), \mathbf{A}(x))$ and $V(\mathbf{x}, q(x))$ is a interaction potential of electrons and phonons. In the Rodriguez-Nuñez (1990) model $V(x, q) = mgq$. For an inhomogeneous material, the quantities ρ and s may be considered as functions of the space points and field variable $q(x)$.

The Lagrangian density (1) is singular in the sense of Dirac. First we determine the constraints of this system in phase space. The canonical momenta associated with the fields ψ, q, A_μ are

$$\pi_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^*, \qquad \pi_{\psi^*} = \frac{\partial\mathcal{L}}{\partial\dot{\psi}^*} = 0$$

$$\pi_q = \frac{\partial\mathcal{L}}{\partial\dot{q}} = \rho\dot{q}, \qquad \pi^\mu = \frac{\partial\mathcal{L}}{\partial\dot{A}_\mu} = -F^{0\mu} \tag{3}$$

respectively. The canonical Hamiltonian density is

$$\mathcal{H}_c = \pi_\psi\dot{\psi} + \pi_{\psi^*}\dot{\psi}^* + \pi_q\dot{q} + \pi^\mu\dot{A}_\mu - \mathcal{L}$$

$$= \frac{1}{2}\pi_i^2 - A^0\partial\pi_i + \frac{1}{4}F_{ik}F_{ik} - e\psi^*A_0\psi$$

$$- \frac{1}{2m}\psi^*[(\nabla - ie\mathbf{A})^2]\psi + \frac{1}{2\rho}\pi_q^2 + \frac{1}{2}(\nabla q)^2 + V\psi^*\psi \tag{4}$$

Then we have three primary constraints

$$\phi_1^0 = \pi_\psi - i\psi^* \approx 0 \tag{5}$$

$$\phi_2^0 = \pi_{\psi^*} \approx 0 \tag{6}$$

$$\phi_3^0 = \pi^0 \approx 0 \tag{7}$$

The total Hamiltonian is

$$H_T = \int d^3x[\mathcal{H}_c + \lambda_1\phi_1^0 + \lambda_2\phi_2^0 + \lambda_3\phi_3^0] \tag{8}$$

where $\lambda_i(x)$ $(i = 1, 2, 3)$ are Lagrange multipliers. The stationarity of the primary constraint ϕ_1^0, $\{\phi_1^0, H_T\} \approx 0$, leads to the equation for determining the Lagrange multiplier λ_2,

$$i\lambda_2 \approx e\psi^* A_0 + \frac{1}{2m}[\nabla^2\psi^* + ie\nabla\psi^*\cdot\mathbf{A} + ie\nabla\cdot(\psi^*\mathbf{A}) - e^2A^2\psi^*] - V\psi^* \tag{9}$$

The stationarity of the primary constraint ϕ_2^0, $\{\phi_2^0, H_T\} \approx 0$, yields

$$i\lambda_1 \approx -e\psi A_0 - \frac{1}{2m}[\nabla^2\psi - ie\nabla\cdot(\psi\mathbf{A}) - ie\mathbf{A}\cdot\nabla\psi - e^2A^2\psi] - V\psi \tag{10}$$

The stationarity of the primary constraint ϕ_3^0, $\{\phi_3^0, H_T\} \approx 0$, leads to the following secondary constraint:

$$\phi^1 = \partial_i\pi_i + e\psi^*\psi \approx 0 \tag{11}$$

The stationarity of the secondary constraint ϕ^1 does not produce any new constraints.

Let us denote $\Lambda_1 = \pi^0 \approx 0$, $\theta_1 = \phi_1^0 \approx 0$, $\theta_2 = \phi_2^0 \approx 0$; one finds a linear combination of the constraints ϕ_1^0, ϕ_2^0, and ϕ^1

$$\begin{aligned}\Lambda_2 &= \phi^1 - ie(\psi\phi_1^0 - \psi^*\phi_2^0)\\ &= \partial_i\pi_i - ie(\psi\pi_\psi - \psi^*\pi_{\psi^*}) \approx 0\end{aligned} \tag{12}$$

It is easy to check that

$$\{\Lambda_1, \Lambda_2\} \approx 0, \qquad \{\Lambda_1, \theta_1\} \approx 0, \qquad \{\Lambda_2, \theta_2\} \approx 0 \tag{13}$$

$$\{\theta_1, \Lambda_2\} = ie\theta_1\delta(\mathbf{x} - \mathbf{y}) \approx 0 \tag{14}$$

$$\{\theta_2, \Lambda_2\} = ie\theta_2\delta(\mathbf{x} - \mathbf{y}) \approx 0 \tag{15}$$

$$\{\theta_1, \theta_2\} = i\delta(\mathbf{x} - \mathbf{y}) \tag{16}$$

Hence, the constraints Λ_1 and Λ_2 are first class, while the constraints θ_1 and θ_2 are second class.

According to path-integral quantization, for each first-class constraint, one must choose a gauge condition. Consider the Coulomb gauge

$$\Omega_2 = \partial_i A_i \approx 0 \tag{17}$$

By the stationarity of Ω_2, $\partial_0\Omega_2 \approx 0$, one has another gauge constraint,

$$\Omega_1 = \partial_i\pi_i + \nabla^2 A_0 \approx 0 \tag{18}$$

The phase-space generating functional of the Green function for a singular Lagrangian can be written as (Senjanovic, 1976)

$$Z[J] = \int \mathcal{D}\phi^\alpha \, \mathcal{D}\pi_\alpha \prod_{i,k,l} \delta(\theta_i)\delta(\Lambda_k)\delta(\Omega_l) \, \det|\{\Lambda_k, \Omega_l\}|$$
$$\times (\det|\{\theta_i, \theta_j\}|)^{1/2} \exp\left\{ i \int d^4x \, (\pi_\alpha \dot{\phi}^\alpha - \mathcal{H}_c + J_\alpha \phi^\alpha) \right\} \quad (19)$$

Using the integral properties of the Grassmann variables $\overline{C}_k(x)$ and $C_l(x)$, we can write the expression (19) as (Li, 1995)

$$Z[J] = \int \mathcal{D}\phi^\alpha \, \mathcal{D}\pi_\alpha \, \mathcal{D}\lambda_m \, \mathcal{D}\overline{C}_k \, \mathcal{D}C_l \exp\left\{ iI^p_{\text{eff}} + i \int d^4x \, J_\alpha \phi^\alpha \right\} \quad (20)$$

where

$$I^p_{\text{eff}} = \int d^4x \, \mathcal{L}^p_{\text{eff}} = \int d^4x \, (\mathcal{L}^p + \mathcal{L}_m + \mathcal{L}_{gh}) \quad (21)$$

$$\mathcal{L}^p = \pi_\alpha \dot{\phi}^\alpha - \mathcal{H}_c \quad (22)$$

$$\mathcal{L}_m = \lambda_j \theta_j + \lambda_k \Lambda_k + \lambda_l \Omega_l \quad (23)$$

$$\mathcal{L}_{gh} = \int d^4y \, [\overline{C}_k(x)\{\Lambda_k(x), \Omega_l(y)\} C_l(y)$$
$$+ \tfrac{1}{2}\overline{C}_i(x)\{\theta_i(x), \theta_j(y)\} C_j(y)] \quad (24)$$

$\lambda_m = (\lambda_j, \lambda_k, \lambda_l)$ are multiplier fields. Here we did not introduce the exterior sources for canonical momenta $\pi_\alpha(x)$. For a system of interacting photons, electrons, and phonons the $\phi^\alpha = (A_\mu, \psi, \psi^*, q)$ in expression (20), and $\pi_\alpha = (\pi^\mu, \pi_\psi, \pi_{\psi^*}, \pi_q)$ are canonical momenta associated with ϕ^α. It is easy to check that the factors $\det|\{\Lambda_k, \Omega_l\}|$ and $\det|\{\theta_i, \theta_j\}|$ for this system are independent of the field variables; thus, one can omit these factors from the generating functional (19), and the expression (19) can be written as

$$Z[J, T, U, V] = \int \mathcal{D}\phi^\alpha \, \mathcal{D}\pi_\alpha \, \mathcal{D}\lambda \, \mathcal{D}\mu \, \mathcal{D}\omega$$
$$\times \exp\left\{ i \int d^4x \, (\mathcal{L}^p_{\text{eff}} + J_\alpha \phi^\alpha + T_k \lambda_k + U_l \omega_l + V_i \mu_i) \right\} \quad (25)$$

where

$$\mathcal{L}^p_{\text{eff}} = \mathcal{L}^p + \lambda_k \Lambda_k + \omega_l \Omega_l + \mu_i \theta_i \quad (26)$$

$$\mathcal{L}^p = \pi_\psi \dot{\psi} + \pi_{\psi^*} \dot{\psi}^* + \pi_q \dot{q} + \pi^\mu \dot{A}_\mu - \mathcal{H}_c \quad (27)$$

$\lambda_k(x)$, $\omega_l(x)$, and $\mu_i(x)$ are multiplier fields connected with the constraints Λ_k,

Ω_l, and θ_i, respectively. In expression (25) we introduced the exterior sources $J_\alpha = (J_\mu, \xi^*, \xi, \eta)$, T_k, U_l, and V_i with respect to the field $\phi^\alpha = (A^\mu, \psi, \psi^*, q)$, λ_k, ω_l, and μ_i, respectively.

For the case where the quantity ρ depends on the coordinate and field variable q, even when the path integral (25) can be carried out by explicit integration over the canonical momenta π_q, the effective Lagrangian is singular with a δ-function (Lee and Yang, 1962). In this case it is more useful to investigate the transformation property of the system under the canonical variables. This we do in the following section.

3. CANONICAL WARD IDENTITIES FOR A SYSTEM WITH SINGULAR LAGRANGIAN

The identities relating the Green function in QED were obtained by Ward (1950) and Takahashi (1957); the generalization of these identities was given by Slavnov (1972) and Taylor (1971). Ward identities and their generalization play an important role in modern field theories. Ward identities have been generalized to supersymmetry (Joglekar, 1991) and superstrings (Danilov, 1991) and other problems. The derivation for Ward identities in the functional integration method is usually discussed by using the configuration-space generating functional (Suura and Young, 1973; Lhallabi, 1989). As is well known, phase-space path integrals are more basic than configuration-space path integrals; the latter provide a Hamiltonian quadratic in canonical momenta, whereas the former apply to arbitrary Hamiltonians (Mizrahi, 1978). Thus, the phase-space form of the path integral is a necessary precursor to the configuration form. In certain integrable cases while the "mass" depends on "coordinates" (Lee and Yang, 1962; Gerstein *et al.*, 1971) or on "coordinates" and momenta (Du *et al.*, 1980), the phase-space generating functional can be simplified by carrying out explicit integration over momenta, and the effective Lagrangians in configuration space are singular with a δ-function. For a constrained Hamiltonian system with complex constraints, it is very difficult or even impossible to carry out the integration over momenta. In these cases the Ward identities cannot be derived via a generating functional with a Lagrangian (or effective Lagrangian) in configuration space as in the traditional treatment. Based on the phase-space generating functional, it is more useful to investigate its transformation properties under the transformation of canonical variables. This problem was developed (Li, 1995) in previous work. Here a slight modification of our previous discussion is presented and some applications to a system of interacting photons, electrons, and phonons are given.

Let us consider an infinitesimal transformation in phase space

$$\begin{cases} \phi^{\alpha\prime}(x) = \phi^\alpha(x) + \delta\phi^\alpha(x) = \phi^\alpha(x) + S^\alpha_\sigma \epsilon^\sigma(x) \\ \pi^{\alpha\prime}(x) = \pi_\alpha(x) + \delta\pi_\alpha(x) = \pi_\alpha(x) + T_{\alpha\sigma}\epsilon^\sigma(x) \end{cases} \tag{28}$$

where $\epsilon^\sigma(x)$ $(\sigma = 1, 2, \ldots, r)$ are infinitesimal arbitrary functions whose values and their derivatives on the boundary of the space-time domain vanish, and S^α_σ and $T_{\alpha\sigma}$ are linear differential operators. Under the transformation (28), the generating functional (20) is invariant. Here we do not introduce exterior sources for canonical momenta (Li, 1995). The variation of the effective canonical action (21) under the transformation (28) is given by (Li, 1993)

$$\delta I^p_{\text{eff}} = \int d^4x \left[\frac{\delta I^p_{\text{eff}}}{\delta\phi^\alpha}\delta\phi^\alpha + \frac{\delta I^p_{\text{eff}}}{\delta\pi_\alpha}\delta\pi_\alpha + \frac{d}{dt}(\pi_\alpha\delta\phi^\alpha)\right] \tag{29}$$

Let it be supposed that the Jacobian of the transformation (28) is equal to unity. From the boundary conditions of $\epsilon^\sigma(x)$ and the invariance of the generating functional (20) under the transformation (28), this leads to $\delta Z/\delta\epsilon^\sigma(x) = 0$. Thus we obtain the following canonical Ward identities:

$$[\tilde{S}^\alpha_\sigma(\delta I^p_{\text{eff}}/\delta\phi^\alpha) + \tilde{T}_{\alpha\sigma}(\delta I^p_{\text{eff}}/\delta\pi_\alpha) + \tilde{S}^\alpha_\sigma J_\alpha]_{\substack{\pi\to\partial\mathcal{L}/\partial\dot\phi^\alpha \\ \phi^\alpha\to(1/i)\delta/\delta J_\alpha}} Z[J] = 0 \tag{30}$$

where $\tilde{S}^\alpha_\sigma$ and $\tilde{T}_{\alpha\sigma}$ are adjoint operators with respect to S^α_σ and $T_{\alpha\sigma}$, respectively (Li, 1987).

Let us now construct the gauge transformation for a system with Lagrangian (1). Dirac (1964) in his work on the generalized canonical formalism conjectured that all first-class constraints are independent generators of the gauge transformation. In spite of the lack of a proof of this conjecture we do not know of any physically important system for which Dirac's conjecture leads to the wrong result. I have shown (Li, 1991) that a system has both first-class and second-class constraints if the series of secondary first-class constraints derived from primary first-class constraints is completely separated from the series of second-class ones; the generator of gauge transformation can be constructed by using all first-class constraints. For a system with Lagrangian (1) belonging to this category, the first-class constraints are (7) and (12), and the second-class constraints are (5) and (6). The gauge generator for this system can be written as (Li, 1991)

$$G = \int d^3x \,\{\dot\epsilon(x)\pi_0(x) - \epsilon(x)[\partial_i\pi_i(x) - ie(\psi(x)\pi_\psi(x) - \psi^*(x)\pi_{\psi^*}(x))]\} \tag{31}$$

This generator produces the following transformation:

$$\begin{cases} \delta A_\mu(x) = \{A_\mu(x), G\} = \partial_\mu \epsilon(x) \\ \delta \pi^\mu(x) = \{\pi^\mu(x), G\} = 0 \\ \delta\psi(x) = \{\psi(x), G\} = ie\epsilon(x)\psi(x) \\ \delta\pi_\psi(x) = \{\pi_\psi(x), G\} = -ie\epsilon(x)\pi_\psi(x) \end{cases} \tag{32}$$

Under the transformation (32) the expression (27) is invariant. The change of (26) under the transformation (32) is given by

$$\delta\mathcal{L}^p_{\text{eff}} = \lambda_2\overline{\Lambda}_2(\psi^*, \pi_{\psi^*})\epsilon(x) + \omega_1\nabla^2\dot{\epsilon}(x) + \omega_2\nabla^2\epsilon(x) \tag{33}$$

where $\overline{\Lambda}_2$ is a function of ψ^* and π_{ψ^*}. The Jacobian of the transformation (32) is equal to unity. The generating functional (25) is invariant under the transformation (32); this yields the following Ward identities:

$$\int \mathcal{D}\phi^\alpha\, \mathcal{D}\pi_\alpha\, \mathcal{D}\lambda\, \mathcal{D}\mu\, \mathcal{D}\omega\, (\lambda_2\overline{\Lambda}_2 - \nabla^2\dot{\omega}_1 + \nabla^2\omega_2 - \partial_\mu J^\mu + \xi^*\psi - \xi\psi + \eta q)$$

$$\times \exp\left[i\int d^4x\, (\mathcal{L}^p_{\text{eff}} + J_\alpha\psi^\alpha + T_k\lambda_k + U_l\omega_l + V_i\nu_i)\right] = 0 \tag{34}$$

or

$$\left(\overline{\Lambda}_2\frac{\delta}{\delta T_2} - \nabla^2\partial_0\frac{\delta}{\delta U_1} + \nabla^2\frac{\delta}{\delta U_2} - \partial_\mu J^\mu + \xi^*\frac{\delta}{\delta\xi^*} - \xi\frac{\delta}{\delta\xi} + \eta\frac{\delta}{\delta\eta}\right)$$

$$\times Z[J, T, U, V] = 0 \tag{35}$$

Let $Z[J_\alpha, T, U, V] = \exp(iW[J_\alpha, T, U, V])$ and use the definition of the generating functional of proper vertices $\Gamma[\phi^\alpha, \lambda, \omega, \mu]$ which is given by performing a functional Legendre transformation on $W[J_\alpha, T, U, V]$,

$$\Gamma[\phi^\alpha, \lambda, \omega, \mu] = W[J_\alpha, T, U, V] - \int d^4x\, (J_\alpha\phi^\alpha + T_k\lambda_k + U_l\omega_l + V_i\mu_i) \tag{36}$$

and

$$\frac{\delta W}{\delta J_\alpha(x)} = \phi^\alpha(x), \qquad \frac{\delta\Gamma}{\delta\phi^\alpha(x)} = -J_\alpha(x)$$

$$\frac{\delta W}{\delta T_k} = \lambda_k(x), \qquad \frac{\delta\Gamma}{\delta\lambda_k(x)} = -T_k(x) \tag{37}$$

$$\frac{\delta W}{\delta U_l(x)} = \omega_l(x), \qquad \frac{\delta \Gamma}{\delta \omega_l(x)} = -U_l(x)$$

$$\frac{\delta W}{\delta V_i(x)} = \mu_i(x), \qquad \frac{\delta \Gamma}{\delta \mu_i(x)} = -V_i(x)$$

Then (35) becomes

$$\lambda_2\overline{\Lambda}_2 - \nabla^2\dot{\omega}_1 + \nabla^2\omega_2 + \partial_\mu\left(\frac{\delta\Gamma}{\delta A_\mu(x)}\right) + \psi(x)\frac{\delta\Gamma}{\delta\psi(x)}$$

$$- \psi^*(x)\frac{\delta\Gamma}{\delta\psi^*(x)} + q(x)\frac{\delta\Gamma}{\delta q(x)} = 0 \tag{38}$$

We functionally differentiate (38) with respect to $\psi(x_2)$ and $\psi^*(x_3)$ and set all fields (including multiplier fields) equal to zero, $A_\mu = \psi = \psi^* = q = \lambda_k = \omega_l = \mu_i = 0$; we obtain

$$\partial^\mu_{x_1}\frac{\delta^3\Gamma[0]}{\delta\psi^*(x_3)\delta\psi(x_2)\delta A^\mu(x_1)}$$

$$= \delta(x_1 - x_2)\frac{\delta^2\Gamma[0]}{\delta\psi(x_1)\delta\psi^*(x_3)}$$

$$- \delta(x_1 - x_3)\frac{\delta^2\Gamma[0]}{\delta\psi^*(x_1)\delta\psi(x_2)} \tag{39}$$

On performing the Fourier transformation of (39), we find

$$q^\mu\Gamma_\mu(p, q, p + q) = S_F^{-1}(p + q) - S_F^{-1}(q) \tag{40}$$

where Γ_μ is a proper vertex for photons and electrons and S_F is a propagator of electrons. We functionally differentiate (38) with respect to $q(x_2)$ and $q(x_3)$ and set all fields equal to zero; we get

$$\partial^\mu_{x_1}\frac{\delta^3\Gamma[0]}{\delta q(x_3)\delta q(x_2)\delta A^\mu(x_1)}$$

$$= \delta(x_1 - x_2)\frac{\delta^2\Gamma[0]}{\delta q(x_1)\delta q(x_3)}$$

$$+ \delta(x_1 - x_3)\frac{\delta^2\Gamma[0]}{\delta q(x_1)\delta q(x_2)} \tag{41}$$

Similarly, differentiating (38) many times with respect to field variables and setting all fields equal to zero, one can obtain various Ward identities for proper vertices.

This formulation for deriving Ward identities of proper vertices has a significant advantage is that one does not need to carry out explicit integration over canonical momenta as one usually does.

4. GLOBAL CANONICAL SYMMETRY AND CONSERVED CHANGE AT THE QUANTUM LEVEL

The connections between global symmetries and conservation laws are usually referred to as Noether theorems in classical theory. Noether theorems are formulated in terms of Lagrange variables in configuration space. The canonical symmetries of a system in phase space for classical theory are investigated in Li (1991). We further study here the connection between global canonical symmetries and conserved charges at the quantum level and give some applications to a system of interacting photons, electrons, and phonons.

For the sake of simplicity, we denote $\phi = (\phi^\alpha, \lambda_m, \overline{C}, C)$, $\pi = (\pi_\alpha)$, and $J = (J_\alpha)$ in expression (20), which thus can be written as

$$Z[J] = \int \mathcal{D}\psi\, \mathcal{D}\pi \exp\left[i \int d^4x\, (\mathcal{L}^p_{\text{eff}} + J\phi)\right] \tag{42}$$

Consider an infinitesimal global transformation in extended phase space

$$\begin{cases} x^{\mu'} = x^\mu + \Delta x^\mu = x^\mu + \epsilon_\sigma \tau^{\mu\sigma}(x, \phi, \pi) \\ \phi'(x') = \phi(x) + \Delta\phi(x) = \phi(x) + \epsilon_\sigma \xi^\sigma(x, \phi, \pi) \\ \pi'(x') = \pi(x) + \Delta\pi(x) = \pi(x) + \epsilon_\sigma \eta^\sigma(x, \phi, \pi) \end{cases} \tag{43}$$

where ϵ_σ ($\sigma = 1, 2, \ldots, r$) are infinitesimal arbitrary parameters, and $\tau^{\mu\sigma}$, ξ^σ, and η^σ are some functions of x, $\phi(x)$, and $\pi(x)$. We suppose that the effective canonical action (21) is invariant under the transformation (43). Now let us further consider the following local transformation connected with the transformation (43):

$$\begin{cases} x^{\mu'} = x^\mu + \Delta x^\mu = x^\mu + \epsilon_\sigma(x) \tau^{\mu\sigma}(x, \phi, \pi) \\ \phi'(x') = \phi(x) + \nabla\phi(x) = \phi(x) + \epsilon_\sigma(x) \xi^\sigma(x, \phi, \pi) \\ \pi'(x') = \pi(x) + \Delta\pi(x) = \pi(x) + \epsilon_\sigma(x) \eta^\sigma(x, \phi, \pi) \end{cases} \tag{44}$$

where $\epsilon_\sigma(x)$ ($\sigma = 1, 2, \ldots, r$) are infinitesimal arbitrary functions whose values and derivatives vanish on the boundary of the space-time domain. Under the transformation (44), the variation of the effective canonical action (21) is given by (Li, 1993)

$$\delta I_{\text{eff}}^{p} = \int d^4x\, \epsilon_\sigma(x)\left\{\frac{\delta I_{\text{eff}}^{p}}{\delta\phi}(\xi^\sigma - \phi_{,\mu}\tau^{\mu\sigma}) + \frac{\delta I_{\text{eff}}^{p}}{\delta\pi}(\eta^\sigma - \phi_{,\mu}\tau^{\mu\sigma})\right.$$
$$\left. + \partial_\mu[(\pi\dot{\phi} - \mathcal{H}_{\text{eff}})\tau^{\mu\sigma}] + D[\pi(\xi^\sigma - \pi_{,\mu}\tau^{\mu\sigma})]\right\}$$
$$+ \int d^4x\, \{(\pi\dot{\phi} - \mathcal{H}_{\text{eff}})\tau^{\mu\sigma}\partial_\mu\epsilon_\sigma(x) + \pi(\xi^\sigma - \phi_{,\mu}\tau^{\mu\sigma})D\epsilon_\sigma(x)\} \quad (45)$$

where $\mathcal{H}_{\text{eff}}$ is an effective Hamiltonian density connected with $\mathcal{L}_{\text{eff}}^{p}$. Owing to the assumption that the effective canonical action (21) is invariant under the transformation (43), the first integral in (45) is equal to zero. According to the boundary conditions of $\epsilon_\sigma(x)$, expression (45) can be written as

$$\delta I_{\text{eff}}^{p} = -\int d^4x\, \epsilon_\sigma(x)\{\partial_\mu[(\pi\dot{\phi} - \mathcal{H}_{\text{eff}})\tau^{\mu\sigma}] + D[\pi(\xi^\sigma - \phi_{,\mu}\tau^{\mu\sigma})]\} \quad (46)$$

Let it be supposed that the Jacobian of the transformation (44) is equal to unity; the invariance of the generating functional (42) under the transformation (44) implies $\delta Z/\delta\epsilon_\sigma(x) = 0$. Substituting (44) and (46) into (42) and functionally differentiating with respect to $\epsilon_\sigma(x)$, one gets

$$\int \mathcal{D}\phi\, \mathcal{D}\pi\, \{\partial_\mu[(\pi\dot{\phi} - \mathcal{H}_{\text{eff}})\tau^{\mu\sigma}] + D[\pi(\xi^\sigma - \phi_{,\mu}\tau^{\mu\sigma}) - M^\sigma\}$$
$$\times \exp\left[i\int d^4x\, (\mathcal{L}_{\text{eff}}^{p} + J\phi)\right] = 0 \quad (47)$$

where

$$M^\sigma = J(\xi^\sigma - \phi_{,\mu}\tau^{\mu\sigma}) \quad (48)$$

Functionally differentiating (47) with respect to $J(x)$ n times, one obtains

$$\int \mathcal{D}\phi\, \mathcal{D}\pi\, (\{\partial_\mu[(\pi\dot{\phi} - \mathcal{H}_{\text{eff}})\tau^{\mu\sigma}] + D[\pi(\xi^\sigma - \phi_{,\mu}\tau^{\mu\sigma})] - M^\sigma\}$$
$$\times \phi(x_1)\cdots\phi(x_n) + i\sum_j \phi(x_1)\cdots\phi(x_{j-1})\phi(x_{j+1})\cdots\phi(x_n)N^\sigma\delta(x - x_j))$$
$$\times \exp\left[i\int d^4x\, (\mathcal{L}_{\text{eff}}^{p} + J\phi)\right] = 0 \quad (49)$$

where

$$N^\sigma = \xi^\sigma - \phi_{,\mu}\tau^{\mu\sigma} \quad (50)$$

Let $J = 0$ in (49); one has

$$\langle 0 | T^*\{\partial_\mu[(\pi\dot{\phi} - \mathcal{H}_{\text{eff}})\tau^{\mu\sigma}] + D[\pi(\xi^\sigma - \phi_{,\mu}\tau^{\mu\sigma})]\}\phi(x_1)\cdots\phi(x_n)|0\rangle$$
$$= -i\sum_j \langle 0|T^*[\phi(x_1)\cdots\phi(x_{j-1})\phi(x_{j+1})\cdots\phi(x_n)N^\sigma]|0\rangle\delta(x - x_j) \quad (51)$$

where the symbol T^* stands for the covariantized T-product (Suura and Young, 1973). Fixing t and letting

$$t_1, t_2, \ldots, t_m \to +\infty, \; t_{m+1}, t_{m+2}, \ldots, t_n \to \infty$$

and using the reduction formula (Young, 1987), we can write expression (51) as

$$\langle \text{out}, m|\{\partial_\mu[(\pi\dot{\phi} - \mathcal{H}_{\text{eff}})\tau^{\mu\sigma}] + D[\pi(\xi^\sigma - \phi_{,\mu}\tau^{\mu\sigma})]\}|n - m, \text{in}\rangle = 0 \quad (52)$$

Since m and n are arbitrary, this implies

$$\partial_\mu[(\pi\dot{\phi} - \mathcal{H}_{\text{eff}})\tau^{\mu\sigma}] + D[\pi(\xi^\sigma - \phi_{,\mu}\tau^{\mu\sigma})] = 0 \quad (53)$$

We now take a cylinder in four-dimensional space, the axis of which is directed along the t axis and whose top and bottom V_1 and V_2 are two likespace hypersurfaces $t = t_1$ and $t = t_2$, respectively. If we assume that the field approaches zero rapidly enough, then taking the integral (53) on this cylinder, from Gauss' theorem in the four-dimensional space, we can neglect the contribution to the boundary term of the infinite cylinder connecting V_1 and V_2. Thus we obtain the conserved charges at the quantum level

$$Q^\sigma = \int d^3x\,[\pi(\xi^\sigma - \phi_{,k}\tau^{k\sigma}) - \mathcal{H}_{\text{eff}}\tau^{0\sigma}] \qquad (\sigma = 1, 2, \ldots, r) \quad (54)$$

Consequently, we obtain the following results: If the effective canonical action (21) is invariant under the global transformation (43) and the Jacobian of the corresponding transformation (44) is equal to unity, then there are some conserved charges (54) for such a system. These results hold for anomaly-free theories. These conserved charges at the quantum level correspond to the classical conservation laws via a canonical Noether theorem (Li, 1993). In general the conserved charges differ from the classical ones arising from canonical symmetries (Li, 1993).

The advantage of the above derivation for quantal conserved charges is that we do not need to carry out explicit integration over the canonical momenta in the phase-space generating functional. In the general case it is not possible to carry out this integration.

For a system with Lagrangian (1) whose effective canonical action is invariant under the spatial translation, the Jacobian of this transformation is equal to unity, and in this case $\tau^{0\sigma} = 0$. From expression (54) we obtain the conserved charge (conservation of momentum)

$$\mathbf{P} = -\int d^3x\,(\pi_\mu \nabla A^\mu + \pi_\psi \nabla \psi + \pi_q \nabla q) \tag{55}$$

With the invariance of time translation, $\tau^{i\sigma} = 0$ ($i = 1, 2, 3$), from expression (54) we obtain the conservation of energy on the constrained hypersurface (including the gauge constraints)

$$E = \int d^3x \left\{\frac{1}{2}\pi_i^2 + \frac{1}{4}F_{ik}F_{ik} - A^0(\partial_i \pi_i + e\psi^*\psi) - \frac{1}{2m}\psi^*[(\nabla - ie\mathbf{A})^2]\psi \right.$$
$$\left. + \frac{1}{2\rho}\pi_q^2 + \frac{1}{2}s(\nabla q)^2 + V\psi^*\psi\right\} \tag{56}$$

Under the transformation of spatial rotation, the Jacobians of the transformation of the vector field A_μ and one-component fields $\psi(x)$, $\psi^*(x)$, and $q(x)$ are equal to unity; in this case $\tau^{0\sigma} = 0$. From (54) and the transformation properties of the fields (Schweber, 1961) we obtain the conservation of angular momentum,

$$M_{jk} = \int d^3x \left\{\pi^\mu\left(x_k \frac{\partial A_\mu}{\partial x_j} - x_j \frac{\partial A_\mu}{\partial x_k}\right) + \pi^\mu(\Sigma_{jk})_{\mu\nu}A^\nu \right.$$
$$\left. + \pi_\psi\left(x_k \frac{\partial \psi}{\partial x_j} - x_j \frac{\partial \psi}{\partial x_k}\right) + \pi_q\left(x_k \frac{\partial q}{\partial x_j} - x_j \frac{\partial q}{\partial x_k}\right)\right\} \tag{57}$$

where

$$(\Sigma_{\rho\sigma})_{\mu\nu} = g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu} \tag{58}$$

The effective canonical action is invariant under the following global gauge transformation:

$$\psi'(x) = e^{-ie\epsilon}\psi(x), \qquad \pi'_\psi(x) = e^{ie\epsilon}\pi_\psi(x) \tag{59}$$

From (54) we obtain the conservation of charge

$$Q = e\int d^3x\,\psi^*(x)\psi(x) \tag{60}$$

It is worthwhile to point out that the above conserved charges (55)–(57) and (60) at the quantum level coincide with classical ones deriving from the canonical Noether theorem (Li, 1993).

5. THE EFFECTIVE LAGRANGIAN IN CONFIGURATION SPACE

Now we construct the configuration-space generating functional for a system with Lagrangian (1). For the case ρ = const in (1), the phase-space path integral (25) can be carried out by explicit integration over momenta π_q without a δ-function (Lee and Yang, 1962). We rewrite (25) as follows:

$$\begin{aligned} Z[J^\mu, \xi^*, \xi, \eta] &= \int \mathcal{D}A_\mu\, \mathcal{D}\pi^\mu\, \mathcal{D}\psi\, \mathcal{D}\pi_\psi\, \mathcal{D}\psi^*\, \mathcal{D}\pi_{\psi^*}\, \mathcal{D}q\, \mathcal{D}\pi_q \\ &\quad \times \prod_{i,k,l} \delta(\theta_i)\delta(\Lambda_k)\delta(\Omega_l)\cdot \exp\left[i \int d^4x\, (\pi^\mu \dot{A}_\mu + \pi_\psi \dot{\psi} + \pi_{\psi^*}\dot{\psi}^* + \pi_q \dot{q} \right. \\ &\quad \left. - \mathcal{H}_c + J^\mu A_\mu + \xi^*\psi + \psi^*\xi + \eta q) \right] \end{aligned} \tag{61}$$

We integrate over A^0, π_0, π_ψ, and π_{ψ^*} in expression (61). Then we represent $\delta[\partial_i\pi_i - ie(\psi\pi_\psi + \psi^*\pi_{\psi^*})]$ in the form of a functional integral

$$\begin{aligned} &\delta[\partial_i\pi_i - ie(\psi\pi_\psi + \psi^*\pi_{\psi^*})] \\ &\quad = \exp\left\{ -i \int d^4x\, A_0[\partial_i\pi_i - ie(\psi\pi_\psi + \psi^*\pi_{\psi^*})]\, \mathcal{D}A_0 \right\} \end{aligned} \tag{62}$$

The remain integral over momenta is of Gaussian type and can be easily calculated. As a result, we obtain the expression for Z in the form of a functional integral in configuration space:

$$\begin{aligned} Z[J^\mu, \xi^*, \xi, \eta] &= \int \mathcal{D}A_\mu\, \mathcal{D}\psi\, \mathcal{D}\psi^*\, \mathcal{D}q\, \delta(\partial_i A_i) \\ &\quad \times \exp\left[i \int d^4x\, (\mathcal{L} + J^\mu A_\mu + \xi^*\psi + \psi^*\xi + \eta q) \right] \end{aligned} \tag{63}$$

or

$$\begin{aligned} Z[J^\mu, \xi^*, \xi, \eta] &= \int \mathcal{D}A_\mu\, \mathcal{D}\psi\, \mathcal{D}\psi^*\, \mathcal{D}q \\ &\quad \times \exp\left[i \int d^4x\, (\mathcal{L}_{\text{eff}} + J^\mu A_\mu + \xi^*\psi + \psi^*\xi + \eta q) \right] \end{aligned} \tag{64}$$

where

$$\mathcal{L}_{\text{eff}} = \mathcal{L} - \frac{1}{2\alpha_0}(\partial_i A_i)^2 \tag{65}$$

$\mathcal{L}$ is given by equation (1), and α_0 is a gauge parameter. The effective Lagrangian (65) differs from the original one because of the existence of the constraints for this system.

Using the traditional trick of the functional integration method (Nash, 1978), one can also derive the Ward identities (39) and (41) from the configuration-space generating functional (64).

As is well known, in the tree diagram approximation the generating functional for proper vertices Γ is equal to the effective action in configuration space

$$\Gamma = \int d^4x\, \mathcal{L}_{\text{eff}} = \int d^4x \left[\mathcal{L} - \frac{1}{2\alpha_0}(\partial_i A_i)^2\right] \tag{66}$$

From expression (66) the Feynman rule for a system with Lagrangian (1) can be derived immediately.

6. CONCLUSIONS AND DISCUSSION

A Lagrangian in (1 + 3)-dimensional space-time which describes the photon–electron–phonon interaction is proposed which is a generalization of Rodriguez-Nuñez' (1990) model. The electromagnetic field is included in the present model. The path-integral quantization of this system is formulated with the aid of the Dirac formalism for a singular Lagrangian and the method of functional integration. The phase-space generating functional of the Green function is deduced. The canonical Ward identities for a system with singular Lagrangian are derived. From these identities, relationships among the Green functions for the present model can be obtained. The conserved charges arising from the global canonical symmetries for a system with singular Lagrangian are also derived at the quantum level. In general, the connection between the canonical symmetries and conservations laws in classical theory is no longer preserved in quantum theory. But in the present model the conserved charges arising from the space-time and internal symmetries at the quantum level coincide with the classical ones. The effective Lagrangian in configuration space for this model have been deduced in the case ρ = const. Thus, the Feynman rule for this system can be derived immediately.

Numerous recent investigations of (1 + 2)-dimensional gauge theories with Chern–Simons terms in the Lagrangian have revealed the occurrence of fractional spin and statistics (Banerjee, 1993, 1994; Kim *et al.*, 1994). They have attracted much attention due to their possible relevance to condensed matter phenomena, especially to the fractional quantum Hall effect and high-T_c superconductivity. In those papers the angular momentum was deduced by using a classical Noether theorem, and the results connected with the fractional spin were analyzed. However, whether those results are valid at the quantum level can be investigated by using our formalism, and work along these lines is in progress.

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REFERENCES

Banerjee, R. (1993). *Physical Review D*, **48**, 2905.
Banerjee, R. (1994). *Nuclear Physics B*, **419**, 611.
Danilov, G. S. (1991). *Physics Letters B*, **257**, 285.
Dirac, P. A. M. (1964). *Lectures in Quantum Mechanics*, Yeshiva University, New York.
Du, T.-S., Yin, H.-C., and Ruan, T.-N. (1980). *Nuclear Physics B*, **164**, 103.
Faddeev, L. D. (1970). *Theoretical Mathematics and Physics*, **1**, 1.
Gerstein, I. S., Jackiw, R., Lee, B. W., and Weinberg, S. (1971). *Physical Review D*, **3**, 2468.
Haken, H. (1976). *Quantum Field Theory of Solids*, North-Holland, Amsterdam.
Joglekar, S. D. (1991). *Physical Review D*, **44**, 3879.
Kim, J. K., Kim, W.-T., and Shin, H. (1994). *Journal of Physics A: Mathematical and General*, **27**, 6067.
Lee, T. D., and Yang, C. N. (1962). *Physical Review*, **128**, 885.
Lhallabi, T. (1989). *International Journal of Theoretical Physics*, **28**, 875.
Li, Z.-P. (1987). *International Journal of Theoretical Physics*, **26**, 853.
Li, Z.-P. (1991). *Journal of Physics A: Mathematical and General*, **24**, 4261.
Li, Z.-P. (1993). Classical and quantal dynamics of the constrained system and their symmetry properties, Beijing Polytechnic University.
Li, Z.-P. (1995). *International Journal of Theoretical Physics*, **34**, 523.
Mizrahi, M. M. (1978). *Journal of Mathematical Physics*, **19**, 298.
Nash, C. (1978). *Relativistic Quantum Fields*, Academic Press, New York.
Rodriguez-Nuñez, J. J. (1990). *International Journal of Theoretical Physics*, **29**, 467.
Schweber, S. S. (1961). *An Introduction to Relativistic Quantum Field Theory*, Harper and Row, New York.
Senjanovic, P. (1976). *Annals of Physics*, **100**, 227.
Slavnov, A. A. (1972). *Theoretical and Mathematical Physics*, **10**, 99.
Suura, H., and Young, B.-L. (1973). *Physical Review D*, **8**, 4353.
Takahashi, Y. (1957). *Nuovo Cimento*, **6**, 370.
Taylor, J. C. (1972). *Nuclear Physics B*, **33**, 436.
Ward, J. C. (1950). *Physical Review*, **77**, 2931.
Young, B.-L. (1987). *Introduction to Quantum Field Theories*, Science Press, Beijing.